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ASYMPTOTICALLY EXACT JOINING CONDITIONS AT THE JUNCTION OF PLATES WITH VERY DIFFERENT CHARACTERISTICS†

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The phenomenon of the boundary layer which occurs when plates are joined is studied. A procedure for deriving the asymptotically exact joining (transmission) conditions which associate the two-dimensional equations for the deformation of the plates along the joining line Γ is developed using the method of matched asymptotic expansions. Two situations are discussed in which these conditions turn out to be non-standard: the bending moment in Γ must disappear and the deflection can undergo a jump (for real values of the physical parameters, the longitudinal displacements and forces as well as the bending and the shearing force always remain continuous). One of the situations (the joining of "thick, soft" and a "thin, rigid" shells) is characteristic of a moving loudspeaker system. The results of a numerical experiment, which confirm the asymptotic analysis of the problem, are presented.

It is well known [1–5, etc.] that the phenomenon of an exponential boundary layer (BL) arises close to the edge of a thin three-dimensional plate or shell. This phenomenon can be neglected [6–8, etc.] when determining the leading terms of the asymptotic form far from the edge, but this cannot be done [1, 9–11, etc.] without an analysis of the BL when effects requiring the lower asymptotic terms to be taken into account are of interest. Using the method of combined asymptotic expansions, the natural requirement of the decay of the boundary layer generates asymptotically exact boundary conditions for the (two-dimensional) system of equations in the theory of shells.

Similar phenomena also occur when some kinds of defects (a welded seam, a rigidity rib, the butting of plates, etc.) are located in a small neighbourhood of the contour Γ in the middle surface of a plate or shell. In this case, an exponential form of the decay of the BL is ensured by the joining (transmission) conditions on the contour Γ [12–14, etc.]. Usually, the joining conditions include the continuity of the transverse and longitudinal displacements and of the shearing and longitudinal forces as well as of the deflection and the bending moment. However, if the defect is characterized by new small parameters (in addition to the relative thickness h of the plate) then, under certain situations, an asymptotic analysis of the boundary layer can lead to non-standard joining conditions and this effect is traced below in the case of a number of problems concerning the butt joining of plates.

The solutions of problems in the theory of elasticity in a strip and in the joining of half-strips are studied in Section 1 on the basis of general results [14]. A procedure for deriving the joining conditions which uses the method of matched asymptotic expansions (MAE) is discussed in Section 2. The joining conditions which are obtained contain a certain symmetric (4×4) matrix M which is an integral characteristic of the joining zone. In cases when the elements M_{jk} are continuously dependent on the additional parameters $\alpha_1, \ldots, \alpha_N$, the standard joining conditions are found to be asymptotically exact. If, however, the M_{jk} increase in an unbounded manner when $\alpha_i \to +0$ then non-standard joining conditions can rise for certain relations between α_i and h. Two real problems are considered in Sections 3 and 4: a plate with an almost continuous crack (the small parameter is the relative thickness of the thin bridge) and the joining of "rigid, thin" and "soft, thick" plates (the parameters are the ratios of the thicknesses and the Young's moduli). Finally, in Section 5, the results of calculations of the spectrum of frequencies of electrodynamic loudspeakers are presented; it is only possible to achieve acceptable agreement with the experimental data using asymptotically exact non-standard joining conditions.

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1. THE MODEL BOUNDARY-LAYER PROBLEM

Suppose that $\Pi = (-l/2, l/2) \times \mathbf{R}$ is a strip made of a homogeneous, isotropic, elastic material with Lamé coefficients λ and μ. We consider the plane and antiplane problems of the theory of elasticity

$$\mu\nabla\cdot\nabla V'(x)+(\lambda+\mu)\nabla\nabla\cdot V'(x)=0,\quad x\in\Pi$$
$$\sigma_{12}(V';x_1,\pm l/2)=\sigma_{22}(V';x_1,\pm l/2)=0,\quad x_1\in\mathbf{R} \tag{1.1}$$

$$\mu\nabla\cdot\nabla V_3(x)=0,\quad x\in\Pi;\quad \mu(\partial V_3/\partial x_2)(x_1,\pm l/2)=0,\quad x_1\in\mathbf{R} \tag{1.2}$$

$V' = (V_1, V_2)$ and V_3 are the displacements and $\sigma(V') = (\sigma_{jk}(V'))$ is the two-dimensional stress tensor. We shall find a basis in the space of polynomial solutions of problem (1.1), (1.2)

$$V^{1'}(x)=(-x_2,x_1),\quad V^{2'}=(0,1),\quad V^{3'}=(1,0),\quad V_3^4=1 \tag{1.3}$$

$$V^{5'}(x)=\frac{1}{D}\left(-x_1x_2,\ \frac{x_1^2}{2}+\frac{\lambda}{\lambda+2\mu}\left[\frac{x_2^2}{2}-\frac{l^2}{24}\right]-\frac{l^2}{60}\frac{11\lambda+12\mu}{\lambda+2\mu}\right)$$

$$V^{6'}(x)=\frac{1}{D}\left(\frac{x_2^2}{2}x_1-\frac{3\lambda+4\mu}{\lambda+2\mu}\frac{x_2^3}{6}-\frac{11\lambda+12\mu}{\lambda+2\mu}l^2\frac{x_1}{40}-\right.$$
$$\left.-\frac{x_1^3}{6}-\frac{\lambda x_1}{\lambda+2\mu}\left(\frac{x_2^2}{2}-\frac{l^2}{24}\right)+\frac{11\lambda+12\mu}{\lambda+2\mu}l^2\frac{x_2}{60}\right) \tag{1.4}$$

$$V^{7'}(x)=\frac{l^2}{12D}\left(x_1,-\frac{\lambda x_2}{\lambda+\mu}\right),\quad V_3^8(x)=\frac{x_1}{\mu l},\quad D=\frac{\mu}{3}l^3\frac{\lambda+\mu}{\lambda+2\mu}$$

The components of the vectors $V^j = (V^{j'}, V_3^j)$, which have been omitted from (1.3) and (1.4), are equal to zero. Formulae (1.3) contain the rigid displacements, while the quantities in (1.4) generate the bending moment, the shearing force and the longitudinal forces, respectively.

Suppose that Ω is a composite plane body which, outside a certain circle $B_R = \{x_1 : |x| < R\}$, coincides with the union of the half-strips $\Pi_+ = (-l_+/2, l_+/2) \times (0, +\infty)$ and $\Pi_- = (-l_-/2, l_-/2) \times (-\infty, 0)$ which are made of materials with Lamé constants λ_+, μ_+ and λ_-, μ_-, respectively. We shall denote the vectors defined by formulae (1.4), in which the symbols λ, μ, l and D are given $\pm$ subscripts, by $V_\pm^j = (V_\pm^{j'}, V_{\pm 3}^j)$.

We consider the solutions $\upsilon = (\upsilon', \upsilon_3)$ of a homogeneous problem in the theory of elasticity in the domain Ω. The number of linearly independent solutions which possess an exponential growth at infinity is equal to eight (half of the product of the number of "exists" from the domain to infinity and the number of analogous solutions of the problem in a strip (see [14, Chapter 5]). Four of these solution are trivial. They are the rigid displacements $V^1, \ldots, V^4$ from (1.3). The further four solutions $\upsilon^1, \ldots, \upsilon^4$ are given by the expansions

$$\upsilon^k(x)=V_-^{4+k}(x)+o(\exp(\delta_- x_1)),\quad x_1\to-\infty$$
$$\upsilon^k(x)=V_+^{4+k}(x)+\sum_{j=1}^{4}M_{kj}V^j(x)+o(\exp(-\delta_+ x_1)),\quad x_1\to+\infty \tag{1.5}$$

Here, $\delta_\pm > 0$ are small numbers, $m = (M_{jk})$ is a partitioned (4×4) matrix which depends on the data for the problem (on Ω, $\lambda_\pm$, $\mu_\pm$ and other data). The matrix M can be interpreted as an integral characteristic of an elastic body which is analogous to the virtual mass tensor [15], the elastic polarization matrix [16], etc. We emphasize that, in the case of dimensionless x_i and l_i, the quantities M_{jk}, by (1.3) and (1.4), have dimensions which are the inverse of the dimension of Young's modulus. It can be shown by checking that the matrix M is symmetric.

2. DETERMINATION OF THE JOINING CONDITIONS

Suppose that a plate Q_h is formed by joining the two strip plates $Q_h^{\pm} = (y, z) \in \mathbf{R}^3 : |z| < hl_{\pm}, H_{\pm} > \pm y_1 > Rh, y_2 \in \mathbf{R}\}$ and that it is rigidly clamped at the edges. It is convenient to describe the structure of the joint using the "fast" variables $x = (x_1, x_2)$ where $x_1 = h^{-1}y_1, x_2 = h^{-1}z$. We shall assume that after changing to x coordinates, the cross-sections $\{(y, z) \in Q_h : y_2 = \text{const}\}$ of the plate are transformed into the domain Ω from Section 1. The magnitude of $H = H_+ + H_-$ is reduced to unity by scaling. All of the linear parameters of the problem then become dimensionless. It is assumed that $H_{\pm}, l_{\pm}, R \gg h$. Finally, we assume that the stressed state of the plate is independent of y_2.

It is well known [1–13, etc.] that far from the edges of the plates $Q_h^{\mp}$, the solution of the three-dimensional problem of the theory of elasticity in Q_h is represented as an asymptotic series with a complex structure of the general term. The main approximation to the three-dimensional displacement field in a plate has the form

$$h^N \sum_{j=1}^{3} \sum_{k=0}^{2\tau_j-1} h^{k-\tau_j} \frac{\partial^k u_j^{\pm}}{\partial y_1^k}(y_1)\Phi_{\pm}^{(k,j)}\left(\frac{z}{h}\right) \tag{2.1}$$

We will now explain the notation used in (2.1). The exponent N in the normalizing factor h^N is determined by the order (with respect to h) of the external loads and, moreover, $\tau_1, \tau_2 = 1, \tau_3 = 2$. The vectors $h^{N-1}u^{\pm\prime} = h^{N-1}(u_1^{\mp}, u_2^{\mp})$ (the principal parts of the longitudinal displacements) satisfy a system of Lamé equations which describes the generalized plane stressed state and the functions $h^{N-2}u_3^{\mp}$ (the principal parts of the deflections of the plates $Q_h^{\mp}$) are the solutions of Germain's equations. Finally

$$\begin{aligned}
&\Phi^{(0,j)} = e^j, \quad j = 1,2,3\\
&\Phi^{(1,1)}(z) = -\lambda(\lambda+2\mu)^{-1} z e^3, \quad \Phi^{(1,2)} = 0, \quad \Phi^{(1,3)}(z) = -z e^1\\
&\Phi^{(2,3)}(z) = \left[\frac{\lambda}{\lambda+2\mu}\left(\frac{z^2}{2} - \frac{l^2}{24}\right) - \frac{l^3}{60}\frac{11\lambda+12\mu}{\lambda+2\mu}\right]e^3\\
&\Phi^{(3,3)}(z) = \left[\frac{3\lambda+4\mu}{\lambda+2\mu}\frac{z^3}{6} - \frac{11\lambda+12\mu}{\lambda+2\mu} l^2 \frac{z}{40}\right]e^1
\end{aligned} \tag{2.2}$$

Here e^j is the unit vector in $\mathbf{R}^3$ and the superscripts $\pm$ are omitted. We recall a fact established in [4, Section 6] and [14, Section 5.6] that the set of linear combinations of vectors (1.3) and (1.4) is identical to a linear shell which has been stretched over the vectors

$$U^{(k,j)}(x) = \sum_{q=0}^{k} \frac{1}{q!} x_1^q \Phi^{(k-q,j)}(x_2), \quad j = 1,2,3; \; k = 0,\ldots,2\tau_j - 1 \tag{2.3}$$

To fix our ideas, we fix $N = 1$ in (2.1) and change to x variables. Expanding $u_j^{\mp}$ in Taylor series and taking account of what has been said concerning the sets (1.3), (1.4) and (2.3), we see that expressions (2.1) are equal to the sums

$$\sum_{n=1}^{4} (a_n^{\pm} V^n(x) + B_n^{\pm} V_{\pm}^{4+n}(x)) \tag{2.4}$$

apart from quantities of smaller orders of magnitude, where

$$a_1^{\pm} = \partial^1 u_3^{\pm}(0), \quad a_2^{\pm} = h^{-1}u_3^{\pm}(0), \quad a_3^{\pm} = u_1^{\pm}(0), \quad a_4^{\pm} = u_2^{\pm}(0) \tag{2.5}$$

$$\begin{aligned}
&b_1^{\pm} = hD_{\pm}\partial^2 u_3^{\pm}(0), \quad b_2^{\pm} = -h^2 D_{\pm}\partial^3 u_3^{\pm}(0)\\
&b_3^{\pm} = 12 l_{\pm}^{-2} h D_{\pm}\partial^2 u_1^{\pm}(0), \quad b_4^{\pm} = l_{\pm}\mu_{\pm} h \partial^1 u_2^{\pm}(0)\\
&\partial^m = \frac{\partial^m}{\partial y_1^m}, \quad D_{\pm} = \frac{1}{3}\mu_{\pm} l_{\pm} \frac{\lambda_{\pm}+\mu_{\pm}}{\lambda_{\pm}+2\mu_{\pm}}
\end{aligned} \tag{2.6}$$

The coefficients from (2.4) (which are functions of the parameter h) have the following physical meaning: the deflection, bending and longitudinal displacements on the edge of the plates are given in (2.5), and the bending moment and the shearing and longitudinal forces are given in (2.6). The reduced ($h = 1$) cylindrical stiffnesses $D_{\pm}$ are used.

In accordance with the algorithm for constructing the asymptotic form of the solutions of elliptic boundary-value problems in thin domains, terms of different orders in the parameter h have to be included in the principal asymptotic term (2.1). The lowest terms in the series for the solution of the three-dimensional problem also possess a similar structure. The subsequent discussion can therefore also be adapted to find the (now inhomogeneous) joining conditions in problems for the lowest terms.

The number of assumptions which have been made at the beginning of this section do not affect the gist of the matter. Suppose, for example, that the functions $u_j^{\pm}$ depend on the variable y_2. On changing to x coordinates, the scale of y_2 is unchanged. The "slow" variable y_2 therefore becomes a parameter of the boundary-layer problem and the dependence of $u^{\pm}$ on y_2 thereby only manifests itself in the lowest terms of the asymptotic series. If, however, the plates are joined along the curve Γ, the "fast" variables $x_1 = h^{-1}n, x_2 = h^{-1}z$ are determined using the local coordinates (n, s), where s is the length of an arc in Γ and $|n|$ is the distance to Γ. The variable s now turns out to be the "slow" variable instead of y_2. It is clear that formula (3.6) for the moments and the forces has to be modified in both cases, but this has no effect on the matching procedure presented below.

The solution of the three-dimensional problem in Q^h is only approximated by the sum (2.1) at a distance from the edges of the plates $Q^h_{\pm}$, and the boundary-layer (BL) phenomenon arises close to these edges. The method of combined asymptotic expansions is usually employed in the theory of shells and plates [1–5, 9–13, etc.] In this method, as in the Vishik–Lyusternik method [17], a solution of the boundary-layer type, which decays exponentially on moving away from the edges and which serves, in particular, to remove any discrepancies remaining in the boundary conditions on the side surfaces of the plates, is added to the smooth type of solution (3.1). In the case of the joining of plates, which is considered here, a solution of the boundary-layer type is found from the problem in the domain Ω (Section 1). However, such a problem cannot always be solved in the class of vector functions which disappear at infinity. It is well known (e.g. see [14, Chapter 5]) that the conditions for the solution to decay at infinity are equivalent to the eight equalities which ensure the orthogonality (in the sense of Betti's formula) of the right-hand sides of the problem to the fields $V^1, \upsilon^1, \ldots, V^4, \upsilon^4$ (see (1.5)). Actually, these eight equalities also constitute the required joining conditions which close the equations which the functions $u_j^{\mp}$ from (2.1) satisfy.

The method of matched asymptotic expansions (MAE), which we shall use to find the matching conditions, due to its great convenience, serves as an alternative to the method which has been described. In the MAE method, a new, inner expansion of the solution of the three-dimensional problem is sought in a small neighbourhood of the edges of the plates (we stress that, in the preceding boundary-layer approach, this was added to a smooth type of solution). As in the boundary-layer approach, the inner expansion is described by the solution V of the problem in Ω. However, now, the requirement of exponential decay at infinity is not imposed on V. A matching condition arises instead of this: the polynomial terms in the asymptotic forms of the field $V(x)$ when $x_1 \to \pm\infty$ are identical to expressions (2.4). We now turn our attention to the fact that both expansions, the outer expansion (2.1) and the inner expansion $V(x)$, correspond to the same three-dimensional field and they must therefore be the same, apart from the lowest terms in a certain "intermediate zone" $ch^{-1/2} < |x_1| < Ch^{-1/2}$ (or, what is the same thing, $ch^{1/2} < |y_1| < Ch^{1/2}$). In the MAE method, this coincidence is also ensured using the matching conditions.

We now construct the solution V of the homogeneous problem in the domain Ω for which the quantities (2.4) turn out to be the asymptotic form when $x_1 \to \pm\infty$. Initially, we consider the "left" infinity ($x_1 \to -\infty$). On recalling what was said in Section 1 and taking account of (1.3)–(1.5), we conclude that the linear combination is the natural candidate for the role of the required solution.

$$V(x) = \sum_{n=1}^{4} (a_n^- V^n(x) + b_n^- \upsilon^n(x)) \tag{2.7}$$

Now, suppose that $x_1 \to +\infty$ and that the superscript "+" in (2.4) is fixed. Returning to the second relation in (2.5) and denoting the exponentially small terms by dots, we see that

$$V(x) = \sum_{n=1}^{4} \left\{ a_n^- V^n(x) + b_n^- \left[V_+^{4+n}(x) + \sum_{j=1}^{4} M_{nj} V^j(x) \right] \right\} + \ldots =$$

$$= \sum_{n=1}^{4} b_n^- V_+^{4+n}(x) + \sum_{j=1}^{4} \left(a_j^- + \sum_{n=1}^{4} b_n^- M_{nj} \right) V^j(x) \ldots \tag{2.8}$$

Once again, we make use of the matching conditions and note that the expression which has been separated out on the right-hand side of (2.8) is the same as the quantity (2.4), where the superscript "+" is only taken when eight relations are satisfied. We write these relations in the vector form

$$a^+ = a^- + Mb^- \tag{2.9}$$

$$b^- = b^+ \tag{2.10}$$

Here, $a^{\pm}$ and $b^{\pm}$ are four-dimensional columns, consisting of the quantities (2.5) and (2.6), and M is the (4×4) matrix of the coefficients of expansions (1.5). Equation (2.1) indicates the continuity of the forces and the moment mentioned in the comment on formulae (2.6) and is the first group of natural joining conditions. Moreover, it appears that relation (2.9) is to be interpreted as the conditions for elastic sealing. However, terms of different orders of smallness occur in term (2.1) of the asymptotic series which has been separated out and, what is more, the columns $a^{\pm}$, $b^{\pm}$ contain heterogeneous elements. An additional asymptotic analysis is therefore required when setting up the second group of joining conditions.

We assume that it is possible to eliminate the small parameter completely on changing to the model problem in Ω. The matrix M is then independent of h and, by virtue of (2.5) and (2.6), the components of the column Mb^- are infinitely small when $h \to 0$, and the components of the columns $a^{\pm}$ are equal to $O(1)$ or $O(h^{-1})$. It is therefore necessary to discard the last term in (2.9) to obtain the second group of joining conditions which express the continuity of the displacements and the deflection

$$a^+ = a^- \tag{2.11}$$

When some of the elements of the matrix M turn out to be large (it is not possible to get rid of the parameters in the model problem), the last term in (2.9) may cease to be small and the limit joining conditions obtained from (2.8) may differ from (2.11). Two such situations are considered in the next two sections.

3. TRANSVERSE CRACKS IN A PLATE

We will now consider a plate with two symmetric transverse surface notches between which there is a thin connecting bridge of width $2\varepsilon h$. We now refer to the notation from Section 1 and put $l = l_{\pm} = 1$, $\lambda_{\pm} = \lambda$, $\mu_{\pm} = \mu$ and

$$\Omega = \Pi \backslash \{x\colon x_1 = 0, |x_2| > \varepsilon\} \tag{3.1}$$

We shall assume that ε is a small parameter and find the asymptotic forms of the elements of the matrix $M = M(\varepsilon)$. Algorithms for constructing the asymptotic expansions of the solutions of problems in domains with thin ligaments have been fully developed [18–22, etc.].

We shall only present the final formulae with comments.

The asymptotic form (when $\varepsilon \to +0$) of the solutions (1.5) of the problem in the domain $\Omega = \Omega(\varepsilon)$, and this also means the asymptotic form of the matrix $M(\varepsilon)$, are sought using the MAE method and constructed from special solutions of the problems in the limiting domains (the existing of the necessary solutions follows, for example, from the general results in [14]). We obtain the first two limiting domains $\Pi_{\pm}$ (half-strips) by putting $\varepsilon = 0$ in (3.1). Yet another limiting domain $\Xi = \mathbf{R}^2 \backslash \{\eta \in \mathbf{R}^2\colon \eta_1 = 0, |\eta_2| \geqslant 1\}$ (a plane with a pair of semi-infinite cuts) arises after changing to the new "fast" variables $\eta = \varepsilon^{-1}x$ and, then, to a zero value of the parameter ε.

We will first give the required solutions of the problem in Ξ. The two linear vector functions $V^0(\eta) = (\eta_1, -\lambda(\lambda + 2\mu)^{-1}\eta_2, 0)$ and $V^{-1}(\eta) = (0, 0, \eta_1)$, corresponding to uniaxial loads along the notches, are added to the rigid displacements $V^1, \ldots, V^4$. Furthermore, six non-polynomial solutions arise which are determined by their behaviour when $|\eta| \to \infty$ in the half-planes $\mathbf{R}^2_{\pm} = \{\eta\colon \pm\eta_1 > 0\}$

$$Y^j(\eta) = \pm V^j(\eta) + O(1) \quad (j = -1, 0, 1) \tag{3.2}$$

$$Y^k(\eta) = \mp T^k(\eta) + O(1) \quad (k = 2, 3, 4) \tag{3.3}$$

$$T^k(\eta) = \gamma_k V^k \ln|\eta| + T^{k0}(|\eta|^{-1}\eta)$$

$$\gamma_2 = \gamma_3 = (\lambda + 3\mu)[2\pi\mu(\lambda + 2\mu)]^{-1}, \quad \gamma_4 = (\pi\mu)^{-1}$$

The displacement fields (3.3) describe the deformation Ξ by the differently directed forces at infinity (T^k are the solutions of the Flaman problem on point forces on the boundary of a half-plane). Yet another solution Y_1 corresponds to the moments $\pm\mu m$ applied to $\mathbf{R}^2_\pm$ at infinity. Explicit formulae can be written out for Y^q which, in particular, enable one to refine expansions (3.2) and (3.3), by replacing $O(1)$ by the residues $O(|\eta|^{-2})$ and with linear combinations of the vectors V^2, V^3, V^4 and $T^{-1} = -\partial_1 T^4$, $T^0 = -\partial_1 T^2$, $T^1 = -\partial_1 T^3 (\partial_1 = \partial/\partial\eta_1)$. Only the inequality $\mu m > 0$ is subsequently necessary.

Apart from the rigid displacements $V^1, \dots, V^4$, the displacement fields generated by the concentrated actions $T^q(x)$ at the point $x = 0$ serve as special solutions $Z^q_\pm (q = -1, \dots, 4)$ of the problems in $\Pi_\pm$. The vectors $Z^{-1}_\pm$ and $Z^0_\pm$ decay exponentially when $|x_1| \to \infty$. The point forces ($q = 2, 3, 4$) and the moment ($q = 1$) are balanced by the load at infinity; in other words, the representations

$$Z^k_\pm(x) = V^{4+k}(x) + o(\exp(\mp\delta_\pm x_1)), \quad x_1 \to \pm\infty$$

which are similar to (1.5), are satisfied.

The asymptotic form of the coefficients $M_{kj}(\varepsilon)$ in formulae (1.5) is found after applying the matching procedure in the zone $|x| = O(\varepsilon^{1/2})$ (or $|\eta| = O(\varepsilon^{-1/2})$)

$$\begin{aligned} &M_{11}(\varepsilon) = 2\varepsilon^{-2}\mu^{-1}m^{-1} + O(\mu^{-1}\varepsilon^{-1}|\ln\varepsilon|) \\ &M_{mn}(\varepsilon) = 2\delta_{m,n}\gamma_n |\ln\varepsilon| + O(\mu^{-1}) \quad (n, m = 2, 3, 4) \\ &M_{12}(\varepsilon) = M_{21}(\varepsilon) = O(\mu^{-1}\varepsilon^{-1}), \quad M_{1p}(\varepsilon) = M_{p1}(\varepsilon) = 0 \quad (p = 3, 4) \end{aligned} \tag{3.4}$$

So, the diagonal elements of the matrix $M(\varepsilon)$ increase without limit when $\varepsilon \to +0$.

Suppose that $\varepsilon \ll h^{1/2}$ and $\varepsilon \gg \exp(-h^{-1})$. Then, by (2.5), (2.6) and (3.4), the quantity $M_{11}(\varepsilon)b_1^-$ predominates in the first row of system (2.9) and all the remaining terms are infinitely small compared with it. We transform this row into the equality $b_1^- = 0$ which, together with (2.6) and (2.10), leads to the formulae

$$D_+\partial^2 u_3^+(0) = 0, \quad D_-\partial^2 u_3^-(0) = 0 \tag{3.5}$$

The remaining rows in (2.9) and (2.10) generate the joining conditions

$$\begin{aligned} &12 l_+^{-2} D_+ \partial^1 u_1^+(0) = 12 l_-^{-2} D_- \partial^1 u_1^-(0) \\ &l_+\mu_+\partial^1 u_2^+(0) = l_-\mu_-\partial^1 u_2^-(0) \\ &D_+\partial^3 u_3^+(0) = D_-\partial^3 u_3^-(0), \quad u_j^+(0) = u_j^-(0) \quad (j = 1, 2, 3) \end{aligned} \tag{3.6}$$

According to the assumption $D_\pm = D$, $l_\pm = l$ and $\mu_\pm = \mu$, that is, the coefficients of the derivatives can be abbreviated. Conditions (3.5) and (3.6) mean that the displacements and forces are continuous, the bending moment is equal to zero along the line of the butt joint and the deflection can undergo a jump.

Now, suppose that the parameters ε and $h^{1/2}$ are comparable in magnitude, that is, the equality $\varepsilon = \varepsilon_0 h^{1/2}$, with a multiplier ε_0 of the order of unity, holds. In view of the symmetry of the domain $\Omega(\varepsilon)$, we have $M_{1p}(\varepsilon) = M_{p1}(\varepsilon) = 0$ ($p = 3, 4$). Relations (3.6), which are complemented by the equalities

$$\begin{aligned} &\partial^2 u_3^+(0) - \partial^2 u_3^-(0) = 0 \\ &\partial^2 u_3^+(0) + \partial^2 u_3^-(0) = 2\varepsilon_0^2 \mu D^{-1} m (\partial^1 u_3^+(0) - \partial^1 u_3^-(0)) \end{aligned} \tag{3.7}$$

therefore follow from (2.9) and (2.10).

Consequently, the bending moment is continuous along the line of the butt joint and a condition which is similar to elastic sealing is satisfied.

The asymptotically exact joining conditions (3.6), (3.5) and (3.6), (3.7) allow of a discontinuity in the deflection. The occurrence of discontinuities in the displacements is due to the quantities $M_{nm}(\varepsilon)$ from (3.4) which have a weak logarithmic growth and become quite large in the case of improbably small values of the parameter ε. Finally, according to (3.4), (2.5) and (2.6) when $h^{1/2} \gg \varepsilon$, it is necessary to discard the last term in (2.9) to arrive at the standard joining conditions (2.10) and (2.11).

In treating the problem in Ω, the possibility of contact between the edges of the cracks was not taken into account. This flaw in the linear formulation of the problem does not arise in the case of angular notches or defects of other types which form thin connecting ligaments with the dimensions of ε and h in the plate. In this case, it follows from [19, 23] that the relation is preserved and this means that the conclusions regarding joining conditions (3.5)–(3.7) still hold.

4. THE JOINING OF PLATES OF DIFFERENT THICKNESS

Suppose that $l_- = 1$, $l_+ = \varepsilon$ and $\Omega(\varepsilon) = \Pi_- \cup \Pi_+(\varepsilon)$, where $\Pi_+(\varepsilon) = \{x \in \mathbf{R}^2: x_1 > 0, |x_2| < \varepsilon/2\}$. The half-strips Π_- and $\Pi_+(\varepsilon)$ are made from homogeneous isotropic materials with Lamé constants λ_-, μ_- and λ_+, μ_+. Conditions of complete contact are specified at the butt joint $\{x: x_1 = 0, |x_2| < \varepsilon/2\}$. We will find the asymptotically exact joining conditions assuming that ε and $\tau = \mu_-/\mu_+$ are small parameters (that is, the material of the "thin" plate Q_h^+ is significantly more rigid than the material of the "thick" plate Q_h^-). Without going into detail, we note that, when $\tau = O(1)$ or $\tau \gg 1$, the natural joining conditions turn out to be asymptotically exact.

The interpretation of $\Omega(\varepsilon)$ as an articulation of singularly degenerate sets enables us to use a well known asymptotic procedure [8, 24–27, etc.] which, on the whole, remains the same as in Section 3 in spite of the divergence in the geometric description of the domain $\Omega(\varepsilon)$ and the presence of an additional parameter τ. The half-strips $\Pi_\pm$ turn out to be the limiting domains (Π_+ is obtained from $\Pi_+(\varepsilon)$ by changing to the fast variables η). In the case of problems on the joining of bodies with differing elastic properties [8, 16, 23, 28, etc.], it is characteristic that there is a disruption of the contact conditions; in this case, the conditions in the displacements are inherited by the softer body. Yet another limiting domain therefore arises, that is, the half-plane $\Xi = \mathbf{R}^2_-$ with restraint conditions at the site of the butt joint with the rigid half-plane Π_+. The sets of special solutions are analogous to those which have been indicated in Section 3; the sole change is the fact that formulae (3.2) and (3.3) are only written for $\eta \in \mathbf{R}^2_-$, that is, when $\eta_1 < 0$. Then, it is only necessary that the quantity, occurring in the expansion of the solution Y^1, $\mu_- m > 0$. It is equal to the moment which has to be applied at infinity to rotate the half-plane Ξ, which is fastened in the segment $\{\eta: \eta_1 = 0, |\eta_2| < 1/2\}$, through unit angle.

After applying the matching procedure in the zone $|x| = O(\varepsilon^{1/2})$, $x_1 < 0$ and taking account of the matching condition at the butt joint $\{x: x_1 = 0, |x_2| < \varepsilon/2\}$, the asymptotic form (when $\varepsilon, \tau \to +0$) of the solutions υ^k and of the coefficients $M_{jk}(\varepsilon, \tau)$ in their expansions (1.5) are found

$$
\begin{aligned}
&M_{11}(\varepsilon,\tau) = \varepsilon^{-2}\mu_-^{-1}m^{-1} + O(\varepsilon^{-2}\tau\mu_-^{-1}) \\
&M_{mn}(\varepsilon,\tau) = \delta_{m,n}\gamma_n^{(-)}|\ln\varepsilon| + O(\tau\mu_-^{-1}) \quad (m,n = 2,3,4) \\
&M_{12}(\varepsilon,\tau) = M_{21}(\varepsilon,\tau) = O(\varepsilon^{-1}\mu_-^{-1}) \\
&M_{1q}(\varepsilon,\tau) = M_{q1}(\varepsilon,\tau) = 0 \quad (q = 3,4)
\end{aligned}
\tag{4.1}
$$

The matrix elements $M(\varepsilon, \tau)$ increase without limit when $\varepsilon \to 0$ which leads to non-standard joining conditions.

We shall only consider the most interesting case when $\tau = O(\varepsilon^3)$ in which the cylindrical stiffnesses of the plates are comparable in order of magnitude. By virtue of (5.1) and (2.6)

$$
\begin{aligned}
&b_1^+(h) = O(h\varepsilon^3\mu_+)\partial^2 u_3^+(0) = O(h\mu_-)\partial^2 u_3^-(0) \\
&b_1^-(h) = O(h\mu_-)\partial^2 u_3^-(0) \\
&M_{11}(\varepsilon,\tau)b_1^-(h) = O(\varepsilon^{-2}h)\partial^2 u_3^-(0)
\end{aligned}
$$

Consequently, as in Section 3, we conclude that, when $h^{1/2} \gg \varepsilon \gg \exp(-h^{-1})$, relations (2.9) and (2.10) generate the non-standard joining conditions (3.5) and (3.6). This fact can be confirmed by means of a mental experiment suggested by L. I. Slepyan. We consider the joining of two beams, one of which is a thin metal beam while the other is a thick rubber-like beam and both beams have their outer ends clamped. If the first beam is rotated around the centre of the joint, then the deformation of the second beam will be localized close to this centre (in view of the smallness of the contact area). Hence, in the case of the second beam, the longitudinal displacement and bending averaged over the thickness turn out to be zero, which also corresponds to a jump in the deflection in the one-dimensional equations.

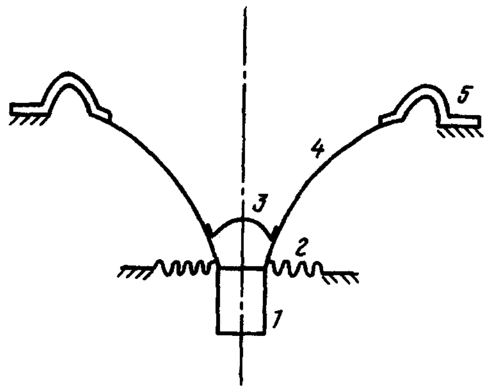

Fig. 1.

At the same time any translational displacement of the right-hand beam leads to a global deformation of the system.

The same non-standard joining conditions (3.5), (3.6) also arise when $D_+ \gg D_-$, $\exp(-h^{-1}) \ll \varepsilon \ll h^{1/2}$. When $\tau = O(\varepsilon^{\alpha})$, $\alpha \in (1, 3)$ (that is, $\varepsilon \ll h^{1/(\alpha-1)}$), non-standard joining conditions are only possible when further constraints are imposed on the smallness of ε: the requirement $\tau = O(\varepsilon^{\alpha})$, $\alpha \in (1, 3)$ must be satisfied. Finally, if $\tau = O(\varepsilon^{\alpha})$ and $\alpha \in (0, 1)$, then the standard conditions (2.10) and (2.11) always appear.

5. A NUMERICAL EXPERIMENT

The conventional moving system of a loudspeaker (shown in Fig. 1 where 1 is an acoustic coil, 2 is an acoustic coil (a cylinder), 3 is a cap, 4 is a diffuser and 5 is a suspension device) is a combination of shells of rotation with strongly differing physicomechanical properties. As a rule, the thicker suspension devices are made of "soft" materials (vulcanized rubber, fabrics, membranes, etc.) while the diffusers are made of "rigid" materials (paper, synthetic membranes, foil, etc.) In this case, the differences in the thicknesses and the Young's moduli can reach values of two and four orders of magnitude respectively.

The following computational scheme has been used in numerous papers (see [29, 30], for example) dealing with the calculation of the natural frequency spectra of loudspeakers: the two dimensional equations of shell theory are solved taking account of the continuity of the displacements and rotations using some numerical method. No such calculations have revealed the low-frequency components of the spectra which are experimentally detected in all types of loudspeakers. Moreover, changing to non-standard joining conditions (dropping the requirement that the angles of rotation should be continuous) we were able to calculate the spectrum of actual loudspeaker constructions using a software package [31] and ensure acceptable agreement between the results and the

Table 1

Element of moving system	Young's modulus N/m^2	Density kg/m^2 x 10	Thickness mm	Poisson's ratio	Natural frequencies	
					A	B
1	1.1×10^{10}	3.83	0.95	0.3	39	331
2	3.0×10^{9}	0.85	0.16	0.3	320	751
3	3.87×10^{8}	1.636	0.6	0.3	627	1402
4	1.7×10^{9}	1.819	0.6	0.3	1007	1495
5	1.56×10^{7}	2.375	0.8	0.4	1418	2006
					1596	2056
					1860	
					2073	

experimental data. The characteristics of one of the constructions are collected together in Table 1 (the numbers of the elements correspond to those shown in the figure). This table shows the first terms of the sequences of natural frequencies calculated taking account of a discontinuity (A) and continuity (B) in the angles of rotation,

The experimental determination of the resonances of this moving system, carried out by measuring the modulus of the total electrical impedance, accurately separates out several of the first resonances at 36 Hz, 630 Hz, etc., which corresponds to the calculated values of the lowest natural frequencies (A). The results presented in column B give a very high frequency for the first resonance. As would be expected, the type of joining conditions has a weak effect on the results of the calculation of the higher natural frequencies.

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